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A two-level atom coupled to a two-dimensional supersymmetric and shape-invariant system: models

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Abstract
A class of bound-state problems which represents the coupling of a two-level atom with a two-dimensional supersymmetric system involving two shape-invariant potentials is introduced. We study two models with different coupling Hamiltonians.

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1. Introduction

Solvable models in quantum theory are so rare that they are worth studying on their own right. Even though they describe oversimplified limiting cases, they could represent salient features of the physical phenomena involved and can be useful in exploring various approximations indispensable for the treatment of more realistic cases. Supersymmetric quantum mechanics together with the shape-invariance concept represents an elegant and powerful method to exactly solve a set of potential systems (such as harmonic oscillator, Coulomb, Morse, Pöschl–Teller, Húlthen, etc) [1].

Supersymmetric quantum mechanics is usually studied in the context of one-dimensional systems [1]. The partner Hamiltonians

$$\hat{H}_- = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(-)}(x) = \hbar\Omega\hat{A}^\dagger\hat{A} \qquad \text{and} \qquad \hat{H}_+ = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(+)}(x) = \hbar\Omega\hat{A}\hat{A}^\dagger \tag{1}$$

can be written in terms of the dimensionless operators

$$\hat{A} \equiv \frac{1}{\sqrt{\hbar\Omega}}\left\{W(x) + \frac{i}{\sqrt{2M}}\hat{p}\right\} \qquad \text{and} \qquad \hat{A}^\dagger \equiv \frac{1}{\sqrt{\hbar\Omega}}\left\{W(x) - \frac{i}{\sqrt{2M}}\hat{p}\right\}, \tag{2}$$

where $\hbar\Omega$ is an energy scale factor and $W(x)$ is the superpotential which is related to the potentials $V^{(\pm)}(x)$ via

$$V^{(\pm)}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2M}} \frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{3}$$

A number of such pairs of Hamiltonians $\hat{H}_\pm$ share an integrability condition called shape invariance [2]. Although not all exactly solvable problems are shape invariant [3], shape invariance, especially in its algebraic formulation [4–6], is a powerful technique to investigate exactly solvable systems.

Relevant progress in the multi-dimensional generalization of the supersymmetric quantum mechanical systems based in the Witten formulation [1] was obtained in [7–9]. In earlier publications [10–12], we introduced a class of shape-invariant coupled-channel problems which generalize the Jaynes–Cummings Hamiltonian [13]. In this paper we study a class of coupled-channel problems consisting of a two-dimensional supersymmetric and shape-invariant system, interacting with a two-level system. We consider two different models which we call direct- and conjugate-coupled models. For each model also we consider two possible forms of coupling: a linear and the other nonlinear in the potential ladder operators.

In section 2 we review fundamentals of the algebraic formulation to shape invariance; in section 3 we present the Hamiltonian for the direct-coupled model involving a two-level atom and a two-dimensional shape-invariant potential system and obtain its eigenstates and eigenvalues; in section 4 we repeat the same for the conjugate-coupled model; in section 5 we apply our generalized results to obtain the eigenvalues and eigenfunctions for three different pairs of shape-invariant potentials (namely two harmonic oscillators, a harmonic oscillator plus a Morse potential and a harmonic oscillator plus a self-similar potential). A conclusion and brief remarks are given in section 6.

2. Algebraic formulation for shape-invariant systems

The Hamiltonian $\hat{H}_-$ of equation (1) is called shape invariant if the condition

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1) \tag{4}$$

is satisfied [2]. The parameter a_2 of the Hamiltonian is a function of its parameter a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. As is written condition (4) does not require the Hamiltonian to be one dimensional, and one does not need to choose the ansatz of equation (2). In the cases studied so far, the parameters a_1 and a_2 are either related by a translation [3, 14] or a scaling [6, 15, 16]. Introducing the parameter translation operator $\hat{T} \equiv \hat{T}(a_1)$ and the similarity transformation $\hat{T}\hat{O}(a_1)\hat{T}^\dagger = \hat{O}(a_2)$ that replace a_1 with a_2 in a given operator [4, 6] and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T} \qquad \text{and} \qquad \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger\hat{A}(a_1), \tag{5}$$

the Hamiltonians of equation (1) take the forms $\hat{H}_- = \hbar\Omega\hat{\mathcal{H}}_-$ and $\hat{H}_+ = \hbar\Omega\hat{T}\hat{\mathcal{H}}_+\hat{T}^\dagger$ where $\hat{\mathcal{H}}_\pm = \hat{B}_\mp\hat{B}_\pm$, and condition (4) can be written as the commutation relation [4] $[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger R(a_1)\hat{T} \equiv R(a_0)$, where we used the identity $R(a_n) = \hat{T}R(a_{n-1})\hat{T}^\dagger$, valid for any $n \in Z$. This commutation relation suggests that $\hat{B}_-$ and $\hat{B}_+$ are the appropriate creation and annihilation operators for the spectra of the shape-invariant potentials provided that their non-commutativity with $R(a_1)$ is taken into account. The additional relations

$$R(a_n)\hat{B}_+ = \hat{B}_+R(a_{n-1}) \qquad \text{and} \qquad R(a_n)\hat{B}_- = \hat{B}_-R(a_{n+1}) \tag{6}$$

readily follow from these results. Since the ground state of the Hamiltonian $\hat{\mathcal{H}}_-$ satisfies the condition $\hat{A}|0\rangle = 0 = \hat{B}_-|0\rangle$, using the relations above it is possible to obtain its normalized nth excited state

$$\hat{\mathcal{H}}_-|n\rangle = e_n|n\rangle \qquad \text{and} \qquad \hat{\mathcal{H}}_+|n\rangle = \{e_n + R(a_0)\}|n\rangle \tag{7}$$

from the ground state $|0\rangle$ using the raising operator $\hat{K}_+$ and the relation [6]

$$|n\rangle = \hat{K}_+^n|0\rangle, \qquad \text{where} \quad \hat{K}_+ = \frac{1}{\sqrt{\hat{\mathcal{H}}_-}}\hat{B}_+. \tag{8}$$

In this case the associated eigenvalues are given by $e_0 = 0$ and

$$e_n = \sum_{k=1}^{n} R(a_k), \qquad \text{for} \quad n \geqslant 1. \tag{9}$$

The action of the $\hat{B}_\pm$ operators on the state given in equation (8) is

$$\hat{B}_+|n\rangle = \sqrt{e_{n+1}}|n+1\rangle \qquad \text{and} \qquad \hat{B}_-|n\rangle = \sqrt{e_{n-1} + R(a_0)}|n-1\rangle \tag{10}$$

and thus [10],

$$\hat{T}\hat{B}_-|n+1\rangle = \sqrt{e_{n+1}}\hat{T}|n\rangle. \tag{11}$$

3. Direct-coupled system

We consider in this study three interacting systems consisting of a single two-level atom or molecule simultaneously coupled with two shape-invariant potential systems $V_x^{(\pm)}(x)$ and $V_y^{(\pm)}(y)$ which are associated with the operators $\hat{A}_x$ and $\hat{A}_y$, respectively. The non-interacting part of our Hamiltonian has a supersymmetric form while its interacting part can assume two possible forms. The first one, called the *direct-coupled system*, contains the interaction in the forms $\hat{A}_x\hat{A}_y$ and $\hat{A}_x^\dagger\hat{A}_y^\dagger$. The second one, called the *conjugate-coupled system* and to be studied in the following section, contains the interaction in the forms $\hat{A}_x\hat{A}_y^\dagger$ and $\hat{A}_x^\dagger\hat{A}_y$. For each model we also consider two possible forms of interaction which correspond to the shape-invariant generalization of the usual and intensity-dependent interaction forms used in optics [17]. The usual interaction Hamiltonian presents a trilinear expression in terms of the atom and the coupling potential operators while for the intensity-dependent case that expression is nonlinear. As shown in [12] for the case of a two-level system coupled to a shape-invariant potential, because of the commensurability of the Rabi frequencies, the intensity-dependent interaction displays periodic revivals in the temporal behaviour of the quantum dynamical variables of the system. Although this also happens with the ordinary interaction models, periodic revivals are created by intensity-dependent interaction stress quantum effects [18].

3.1. Hamiltonian

In treating a two-level system with a lower state $|-\rangle$ and an upper state $|+\rangle$ we can introduce the excitation $\hat{\sigma}_+ \equiv |+\rangle\langle-|$ and de-excitation $\hat{\sigma}_- \equiv |-\rangle\langle+|$ operators as well as the inversion operator $\hat{\sigma}_3 \equiv |+\rangle\langle+| - |-\rangle\langle-|$ which satisfy the commutation relations $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_3$ and $[\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$. By assuming a two-dimensional spinor representation for the eigenstates of the atomic system

$$\chi_- \equiv \langle\chi|-\rangle = \begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad \chi_+ \equiv \langle\chi|+\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \tag{12}$$

it is straightforward to verify that these operators will be represented by matrices

$$\hat{\boldsymbol{\sigma}}_+ = \chi_+\chi_-^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{\sigma}}_- = \chi_-\chi_+^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{\sigma}}_3 = \chi_+\chi_+^\dagger - \chi_-\chi_-^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{13}$$

Thus if we define the polarization matrices $\hat{\boldsymbol{\sigma}}_1 \equiv \hat{\boldsymbol{\sigma}}_- + \hat{\boldsymbol{\sigma}}_+$ and $\hat{\boldsymbol{\sigma}}_2 \equiv i(\hat{\boldsymbol{\sigma}}_- - \hat{\boldsymbol{\sigma}}_+)$ we obtain the Pauli matrices $\hat{\boldsymbol{\sigma}}_i$, for $i = 1, 2$ and 3. We introduce the direct model Hamiltonian which describes the coupling of a two-level system with two shape-invariant potentials $V_x^{(\pm)}(x)$ and $V_y^{(\pm)}(y)$ as

$$\hat{H} = \hat{H}_0 + \hat{H}_\xi, \qquad \text{where} \qquad \hat{H}_0 = \hbar\Omega\left\{\left(\hat{A}_x\hat{A}_x^\dagger + \hat{A}_y\hat{A}_y^\dagger\right)\hat{\sigma}_+\hat{\sigma}_- + \left(\hat{A}_x^\dagger\hat{A}_x + \hat{A}_y^\dagger\hat{A}_y\right)\hat{\sigma}_-\hat{\sigma}_+\right\}, \tag{14}$$

and the operators $\hat{A}_{x,y}$ and $\hat{A}_{x,y}^\dagger$ separately satisfy the shape-invariance condition of equation (4). We consider the interaction Hamiltonian in two possible forms. The forms to be assumed for this Hamiltonian for an usual and a nonlinear interaction, specified respectively when $\xi = \mathrm{U}$ and $\xi = \mathrm{N}$, are given by

$$\hat{H}_\xi = \hbar\Delta\hat{\sigma}_3 + \hbar g \begin{cases} \hat{A}_x\hat{A}_y\hat{\sigma}_+ + \hat{A}_x^\dagger\hat{A}_y^\dagger\hat{\sigma}_-, & \xi = \mathrm{U}, \\ \hat{A}_x\hat{A}_y\sqrt{\hat{N}_x\hat{N}_y}\hat{\sigma}_+ + \sqrt{\hat{N}_x\hat{N}_y}\hat{A}_x^\dagger\hat{A}_y^\dagger\hat{\sigma}_-, & \xi = \mathrm{N}, \end{cases} \tag{15}$$

where g is a real coupling constant strength, Δ is the detuning factor and $\hat{N}_{x,y} = \hat{A}_{x,y}^\dagger\hat{A}_{x,y}$. The interaction Hamiltonian $\hat{H}_\xi$ is responsible for the process of *heating* and *cooling* of the coupled system. It is worth noting that the two-level system is the mechanism that permits the potentials $V_x^{(\pm)}(x)$ and $V_y^{(\pm)}(y)$ to interact. As can be easily checked if we take these potentials to be harmonic oscillators, we obtain the usual and intensity-dependent versions of the Jaynes–Cummings model describing a two-level atom interacting non-resonantly with a two-mode cavity field.

The algebraic formulation presented in section 2 can be applied in the Hamiltonian (14) by using the $\hat{B}_\pm^{(x,y)}$ operators defined by equations (5) with the introduction of the parameter translation operators $\hat{T}_x \equiv \hat{T}_x\left(a_1^{(x)}\right)$ and $\hat{T}_y \equiv \hat{T}_y\left(a_1^{(y)}\right)$ for each shape-invariant potential. Taking into account that the commutation relations $\left[\hat{B}_\mp^{(\alpha)}, \hat{B}_\pm^{(\beta)}\right] = \pm R_\alpha\left(a_0^{(\alpha)}\right)\delta_{\alpha\beta}$ and $\left[\hat{B}_\pm^{(\alpha)}, \hat{B}_\pm^{(\beta)}\right] = 0$ are satisfied, where $R_x\left(a_n^{(x)}\right)$ and $R_y\left(a_n^{(y)}\right)$ are the remainders related to the potentials $V_x^{(\pm)}(x)$ and $V_y^{(\pm)}(y)$, respectively, the final result can be written as $\hat{H} = \hat{\mathcal{T}}_\pm\hat{h}\hat{\mathcal{T}}_\pm^\dagger$ if we define the parameter translation inclusive operator $\hat{\mathcal{T}}_\pm = \hat{T}_x\hat{T}_y\hat{\sigma}_+\hat{\sigma}_- \pm \hat{\sigma}_-\hat{\sigma}_+$ and decompose the Hamiltonian $\hat{h}$ as $\hat{h} = \hat{h}_0 + \hat{h}_\xi$ where

$$\hat{h}_0 = \hbar\Omega\left\{\left(\hat{\mathcal{H}}_+^{(x)} + \hat{\mathcal{H}}_+^{(y)}\right)\hat{\sigma}_+\hat{\sigma}_- + (\hat{\mathcal{H}}_-^{(x)} + \hat{\mathcal{H}}_-^{(y)})\hat{\sigma}_-\hat{\sigma}_+\right\}, \tag{16}$$

and the Hamiltonian $\hat{h}_\xi$ for the two kinds of interaction is given by

$$\hat{h}_\xi = \hbar\Delta\hat{\sigma}_3 \pm \hbar g \begin{cases} \left(\hat{B}_-^{(x)}\hat{B}_-^{(y)}\hat{\sigma}_+ + \hat{B}_+^{(x)}\hat{B}_+^{(y)}\hat{\sigma}_-\right), & \xi = \mathrm{U}, \\ \left(\hat{B}_-^{(x)}\hat{B}_-^{(y)}\sqrt{\hat{\mathcal{H}}_-^{(x)}\hat{\mathcal{H}}_-^{(y)}}\hat{\sigma}_+ + \sqrt{\hat{\mathcal{H}}_-^{(x)}\hat{\mathcal{H}}_-^{(y)}}\hat{B}_+^{(x)}\hat{B}_+^{(y)}\hat{\sigma}_-\right), & \xi = \mathrm{N}. \end{cases} \tag{17}$$

Here we used the fact that $\hat{N}_{x,y} = \hat{B}_+^{(x,y)}\hat{B}_-^{(x,y)} = \hat{\mathcal{H}}_-^{(x,y)}$ and definition (14) together with the unitarity property $\hat{T}_{x,y}^\dagger\hat{T}_{x,y} = \hat{T}_{x,y}\hat{T}_{x,y}^\dagger = \hat{1}$. Note the freedom of sign choice in $\hat{\mathcal{T}}_\pm$ permitted by the form of $\hat{H}_0$.

3.2. Model superalgebra

Defining the supercharge operators related to each shape-invariant potential $\hat{\mathcal{Q}}_\alpha = \hat{B}_-^{(\alpha)}\hat{\sigma}_+$ and their Hermitian adjoint operators $\hat{\mathcal{Q}}_\alpha^\dagger = \hat{B}_+^{(\alpha)}\hat{\sigma}_-$, where $\alpha = x$ or y, it is possible to rewrite the Hamiltonian $\hat{h}_0$ in (16) as

$$\hat{h}_0 = \hbar\Omega\left(\left\{\hat{\mathcal{Q}}_x, \hat{\mathcal{Q}}_x^\dagger\right\} + \left\{\hat{\mathcal{Q}}_y, \hat{\mathcal{Q}}_y^\dagger\right\}\right), \tag{18}$$

and we can verify the commutation and anti-commutation relations

$$[\hat{\mathcal{Q}}_\alpha, \hat{h}_0] = \left[\hat{\mathcal{Q}}_\alpha^\dagger, \hat{h}_0\right] = 0 \qquad \text{and} \qquad \{\hat{\mathcal{Q}}_\alpha, \hat{\mathcal{Q}}_\alpha\} = \left\{\hat{\mathcal{Q}}_\alpha^\dagger, \hat{\mathcal{Q}}_\alpha^\dagger\right\} = 0, \qquad \alpha = x \text{ or } y. \tag{19}$$

The set of relations (18) and (19) characterize the supersymmetry of the Hamiltonian $\hat{h}_0$, with the operators $\hat{\mathcal{Q}}_\alpha$ and $\hat{\mathcal{Q}}_\alpha^\dagger$ as its generators. The commutation relations of (19) state the invariance of the Hamiltonian under this symmetry while the anti-commutation relations express the fermionic character of the supercharge operators. Equation (18) closes the graded Lie algebra with the anti-commutators of $\hat{\mathcal{Q}}_\alpha$ with $\hat{\mathcal{Q}}_\alpha^\dagger$. The supercharges $\hat{\mathcal{Q}}_\alpha$ are interpreted as the operators which change bosonic degrees of freedom into fermionic ones and vice versa.

On the other hand, if we introduce the *hybrid* supercharge operator $\hat{\mathcal{Q}}_{xy} = \hat{B}_-^{(x)}\hat{B}_-^{(y)}\hat{\sigma}_+$ and its adjoint operator $\hat{\mathcal{Q}}_{xy}^\dagger = \hat{B}_+^{(x)}\hat{B}_+^{(y)}\hat{\sigma}_-$, it is possible to rewrite the interaction Hamiltonian (17) for the $\xi = \mathrm{U}$ case as

$$\hat{h}_{\mathrm{U}} = \hbar\left\{\Delta\hat{\sigma}_3 \pm g\left(\hat{\mathcal{Q}}_{xy} + \hat{\mathcal{Q}}_{xy}^\dagger\right)\right\}. \tag{20}$$

Besides that using the definition of $\hat{\mathcal{Q}}_{xy}$, it is easy to verify the additional commutation and anti-commutation relations

$$[\hat{\mathcal{Q}}_{xy}, \hat{h}_0] = \left[\hat{\mathcal{Q}}_{xy}^\dagger, \hat{h}_0\right] = 0 \qquad \text{and} \qquad \{\hat{\mathcal{Q}}_{xy}, \hat{\mathcal{Q}}_{xy}\} = \left\{\hat{\mathcal{Q}}_{xy}^\dagger, \hat{\mathcal{Q}}_{xy}^\dagger\right\} = 0. \tag{21}$$

The commutators imply that $[\hat{h}_0, \hat{h}_{\mathrm{U}}] = 0$ while the anti-commutators express the fermionic character of the hybrid supercharge operators.

In the $\xi = \mathrm{N}$ interaction case, the hybrid supercharge operator and its adjoint operator must be redefined as $\hat{\mathcal{Q}}_{xy} = \hat{B}_-^{(x)}\hat{B}_-^{(y)}\sqrt{\hat{\mathcal{H}}_-^{(x)}\hat{\mathcal{H}}_-^{(y)}}\hat{\sigma}_+$ and $\hat{\mathcal{Q}}_{xy}^\dagger = \sqrt{\hat{\mathcal{H}}_-^{(x)}\hat{\mathcal{H}}_-^{(y)}}\hat{B}_+^{(x)}\hat{B}_+^{(y)}\hat{\sigma}_-$, and expressions (20) and (21) are still valid.

The hybrid supercharge operator $\hat{\mathcal{Q}}_{xy}$ and its adjoint operator $\hat{\mathcal{Q}}_{xy}^\dagger$ are responsible, respectively, for the *heating* and *cooling* process of the coupled system. In the heating process, schematically illustrated in figure 1, the two-level system is excited while the two shape-invariant potentials exhibit a de-excitation process. The inverse *cooling* process of the coupled system is also illustrated in that figure.

3.3. Eigenstates and eigenvalues

Since $\hat{H}_0$ and $\hat{H}_\xi$ commute, it is possible to find a common set of eigenstates for them. To obtain the eigenstates of $\hat{H}_0|\Psi\rangle = E^{(0)}|\Psi\rangle$, we first introduce the dressed states

$$\begin{aligned}|\Psi^{(\pm)}\rangle &= \hat{\mathcal{T}}_\pm\left\{C_{n_xn_y}^{(\pm)}|n_x\rangle_x|n_y\rangle_y|+\rangle + C_{m_xm_y}^{(\pm)}|m_x\rangle_x|m_y\rangle_y|-\rangle\right\}\\ &= \hat{T}_x\hat{T}_yC_{n_xn_y}^{(\pm)}|n_x\rangle_x|n_y\rangle_y|+\rangle \pm C_{m_xm_y}^{(\pm)}|m_x\rangle_x|m_y\rangle_y|-\rangle,\end{aligned} \tag{22}$$

where $C_{\mu\nu}^{(\pm)} \equiv C_{\mu\nu}^{(\pm)}\left[a_1^{(x)}, a_2^{(x)}, \ldots; a_1^{(y)}, a_2^{(y)}, \ldots\right]$ are auxiliary normalization coefficients which depend on the potentials parameters $a_n^{(\alpha)}$. The states $|\nu\rangle_\alpha$ are the eigenstates (8) of the Hamiltonians $\hat{\mathcal{H}}_-^{(\alpha)}$ with eigenvalues $e_0^{(\alpha)} = 0$ and

$$e_\nu^{(\alpha)} = \sum_{j=1}^{\nu} R_\alpha\left(a_j^{(\alpha)}\right), \qquad \text{for} \quad \nu \geqslant 1 \quad \text{and} \quad \alpha = x \text{ or } y. \tag{23}$$

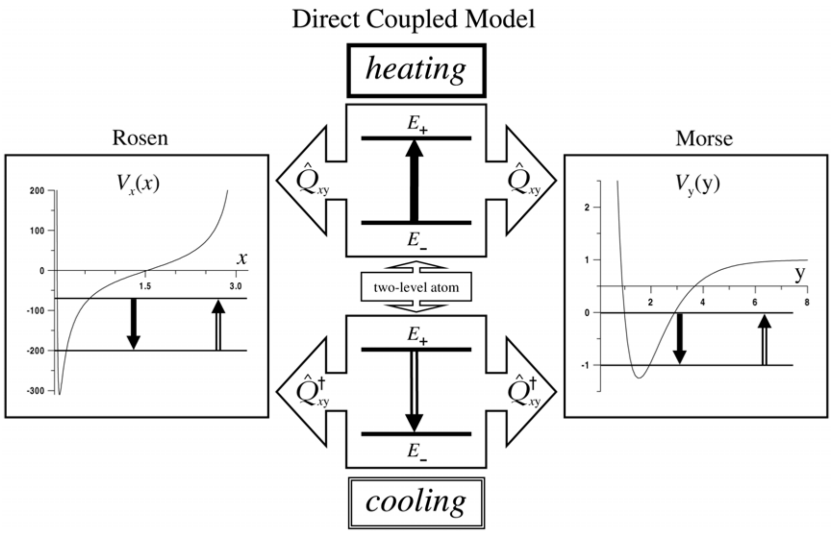

Figure 1. Schematic representation of the *heating* and *cooling* process produced with the action of the *hybrid* supercharge operator $\hat{\mathcal{Q}}_{xy}$ in the case of the direct-coupled model.

Therefore using equations (7), (16) and (22) in the eigenvalue equation and taking into account the commutation between the C-coefficients and the Hamiltonians $\hat{\mathcal{H}}^{(\alpha)}_{\pm}$, the operators action (8), and the unitarity of the operators $\hat{T}_\alpha$ we obtain the system of equations

$$\begin{cases} \left\{\hbar\Omega \hat{T}_x \hat{T}_y \left[e^{(x)}_{n_x} + R_x\left(a^{(x)}_0\right) + e^{(y)}_{n_y} + R_y\left(a^{(y)}_0\right)\right] \hat{T}^\dagger_y \hat{T}^\dagger_x\right\} \hat{T}_x \hat{T}_y C^{(\pm)}_{n_x n_y} |n_x\rangle_x |n_y\rangle_y \\ \quad = E^{(0)} \hat{T}_x \hat{T}_y C^{(\pm)}_{n_x n_y} |n_x\rangle_x |n_y\rangle_y \\ \hbar\Omega\left[e^{(x)}_{m_x} + e^{(y)}_{m_y}\right] C^{(\pm)}_{m_x m_y} |m_x\rangle_x |m_y\rangle_y = E^{(0)} C^{(\pm)}_{m_x m_y} |m_x\rangle_x |m_y\rangle_y. \end{cases} \tag{24}$$

Comparing the coupling potential eigenstates $|\nu\rangle_\alpha$ in the two equations of (24), we conclude that we must have $m_x = n_x + 1$ and $m_y = n_y + 1$ and thus we find that

$$\left|\Psi^{(\pm)}_{n_x n_y}\right\rangle = \hat{T}_x \hat{T}_y C^{(\pm)}_{n_x n_y} |n_x\rangle_x |n_y\rangle_y |+\rangle \pm C^{(\pm)}_{n_x+1, n_y+1} |n_x + 1\rangle_x |n_y + 1\rangle_y |-\rangle \tag{25}$$

and $E^{(0)}_\alpha \equiv E^{(0)}_{n_x n_y} = \hbar\Omega\left(e^{(x)}_{n_x+1} + e^{(y)}_{n_y+1}\right)$. In a two-dimensional spinor representation, the eigenstates (25) of the coupled system can be written in terms of the coupling potential eigenfunctions $\psi^{(x)}_\mu(x)$ and $\psi^{(y)}_\nu(y)$ as

$$\Psi^{(\pm)}_{n_x n_y}(x, y) = \left\langle x, y \middle| \Psi^{(\pm)}_{n_x n_y}\right\rangle = \begin{bmatrix} \hat{T}_x \hat{T}_y C^{(\pm)}_{n_x n_y} \psi_{n_x n_y}(x, y) \\ \pm C^{(\pm)}_{n_x+1, n_y+1} \psi_{n_x+1, n_y+1}(x, y) \end{bmatrix}, \tag{26}$$

where $\psi_{\mu\nu}(x, y) \equiv \langle x|\mu\rangle_x \langle y|\nu\rangle_y = \psi^{(x)}_\mu(x)\psi^{(y)}_\nu(y)$. Note that the orthonormality of the eigenstates $\left|\Psi^{(\pm)}_{n_x n_y}\right\rangle$ imply the following relations among the C-coefficients:

$$\left[C^{(\pm)}_{n_x n_y}\right]^2 + \left[C^{(\pm)}_{n_x+1, n_y+1}\right]^2 = 1 \qquad \text{and} \qquad C^{(\pm)}_{n_x n_y} C^{(\mp)}_{n_x n_y} - C^{(\pm)}_{n_x+1, n_y+1} C^{(\mp)}_{n_x+1, n_y+1} = 0. \tag{27}$$

To determine the eigenvalues of $\hat{H}_\xi$, we need to calculate $\hat{H}_\xi\left|\Psi^{(\pm)}_{n_xn_y}\right\rangle = \mathcal{E}^{(\pm)}_{\xi n_xn_y}\left|\Psi^{(\pm)}_{n_xn_y}\right\rangle$. Using equations (14), (17) and (25) in this eigenvalue equation for the $\xi = \mathrm{U}$ interaction case, we obtain the system of equations

$$\begin{cases} \hbar\Delta\hat{T}_x\hat{T}_yC^{(\pm)}_{n_xn_y}|n_x\rangle_x|n_y\rangle_y \pm \hbar g\hat{T}_x\hat{T}_y\hat{B}^{(x)}_-\hat{B}^{(y)}_-C^{(\pm)}_{n_x+1,n_y+1}|n_x+1\rangle_x|n_y+1\rangle_y \\ \quad = \mathcal{E}^{(\pm)}_{\mathrm{U}n_xn_y}\hat{T}_x\hat{T}_yC^{(\pm)}_{n_xn_y}|n_x\rangle_x|n_y\rangle_y \\ \hbar g\hat{B}^{(x)}_+\hat{B}^{(y)}_+C^{(\pm)}_{n_xn_y}|n_x\rangle_x|n_y\rangle_y \mp \hbar\Delta C^{(\pm)}_{n_x+1,n_y+1}|n_x+1\rangle_x|n_y+1\rangle_y \\ \quad = \pm\mathcal{E}^{(\pm)}_{\mathrm{U}n_xn_y}C^{(\pm)}_{n_x+1,n_y+1}|n_x+1\rangle_x|n_y+1\rangle_y \end{cases} \tag{28}$$

from which, using relations (10) and (11), we find the eigenvalue

$$\mathcal{E}^{(\pm)}_{\mathrm{U}n_xn_y} = \pm\hbar\sqrt{g^2e^{(x)}_{n_x+1}e^{(y)}_{n_y+1} + \Delta^2} \tag{29}$$

and the additional relation

$$C^{(\pm)}_{n_x+1,n_y+1} = \left\{\sqrt{1+\upsilon^2_{n_xn_y}} \mp \upsilon_{n_xn_y}\right\}\left(\hat{T}_x\hat{T}_yC^{(\pm)}_{n_xn_y}\hat{T}^\dagger_y\hat{T}^\dagger_x\right), \qquad \text{where} \quad \upsilon_{n_xn_y} = \frac{\Delta}{g\sqrt{e^{(x)}_{n_x+1}e^{(y)}_{n_y+1}}}. \tag{30}$$

On the other hand, using equations (14), (17) and (25) in the eigenvalue equation for the $\xi = \mathrm{N}$ interaction case and following the same steps used before we obtain the system of equations

$$\begin{cases} \hbar\Delta\hat{T}_x\hat{T}_yC^{(\pm)}_{n_xn_y}|n_x\rangle_x|n_y\rangle_y \pm \hbar g\hat{T}_x\hat{T}_y\hat{B}^{(x)}_-\hat{B}^{(y)}_-\sqrt{\hat{N}_x\hat{N}_y}C^{(\pm)}_{n_x+1,n_y+1}|n_x+1\rangle_x|n_y+1\rangle_y \\ \quad = \mathcal{E}^{(\pm)}_{\mathrm{N}n_xn_y}\hat{T}_x\hat{T}_yC^{(\pm)}_{n_xn_y}|n_x\rangle_x|n_y\rangle_y \\ \hbar g\sqrt{\hat{N}_x\hat{N}_y}\hat{B}^{(x)}_+\hat{B}^{(y)}_+C^{(\pm)}_{n_xn_y}|n_x\rangle_x|n_y\rangle_y \mp \hbar\Delta C^{(\pm)}_{n_x+1,n_y+1}|n_x+1\rangle_x|n_y+1\rangle_y \\ \quad = \pm\mathcal{E}^{(\pm)}_{\mathrm{N}n_xn_y}C^{(\pm)}_{n_x+1,n_y+1}|n_x+1\rangle_x|n_y+1\rangle_y \end{cases} \tag{31}$$

which, in this case, gives the eigenvalue

$$\mathcal{E}^{(\pm)}_{\mathrm{N}n_xn_y} = \pm\hbar\sqrt{\left\{ge^{(x)}_{n_x+1}e^{(y)}_{n_y+1}\right\}^2 + \Delta^2} \tag{32}$$

and an additional relation among the C-coefficients with the same form (30), but with the factor $\upsilon_{n_xn_y}$ now given by $\upsilon_{n_xn_y} = \Delta\big/\left(ge^{(x)}_{n_x+1}e^{(y)}_{n_y+1}\right)$.

To conclude this section, we observe that in both interaction cases the energy eigenvalues of the coupled system $E^{(\pm)}_{\xi n_xn_y} = E^{(0)}_{n_xn_y} + \mathcal{E}^{(\pm)}_{\xi n_xn_y}$ can be written in a dimensionless form as

$$E^{(\pm)}_{\xi n_xn_y}\big/\hbar\Omega = \left\{e^{(x)}_{n_x+1} + e^{(y)}_{n_y+1}\right\} \pm \epsilon\begin{cases}\sqrt{e^{(x)}_{n_x+1}e^{(y)}_{n_y+1} + \delta^2} & \text{if} \quad \xi = \mathrm{U}, \\ \sqrt{\left\{e^{(x)}_{n_x+1}e^{(y)}_{n_y+1}\right\}^2 + \delta^2} & \text{if} \quad \xi = \mathrm{N}, \end{cases} \tag{33}$$

where $\epsilon = g/\Omega$ and $\delta = \Delta/g$. On the other hand, using relation (8), it is possible to obtain the excited states of the coupled system $\left|\Psi^{(\pm)}_{n_xn_y}\right\rangle$ from the two-dimensional spinor state

$$\left|\Psi^{(\pm)}_{00}\right\rangle = \begin{bmatrix}\hat{T}_x\hat{T}_yC^{(\pm)}_{00}|0\rangle_x|0\rangle_y \\ \pm C^{(\pm)}_{11}|1\rangle_x|1\rangle_y\end{bmatrix} \tag{34}$$

by using the expression $\left|\Psi^{(\pm)}_{n_xn_y}\right\rangle = \hat{\mathbf{K}}^{(\pm)}_{n_xn_y}\left|\Psi^{(\pm)}_{00}\right\rangle$, where the raising operator matrix is given by

$$\hat{\mathbf{K}}^{(\pm)}_{n_xn_y} = \hat{\mathbf{T}}_\pm\hat{\mathbf{C}}^{(\pm)}_{n_xn_y}\left\{\hat{K}^{(x)}_+\right\}^{n_x}\left\{\hat{K}^{(y)}_+\right\}^{n_y}\left\{\hat{\mathbf{C}}^{(\pm)}_{00}\right\}^{-1}\hat{\mathbf{T}}^\dagger_\pm \qquad \text{with}$$

$$\begin{cases}\hat{\mathbf{T}}_\pm = \hat{T}_x\hat{T}_y\hat{\boldsymbol{\sigma}}_+\hat{\boldsymbol{\sigma}}_- \pm \hat{\boldsymbol{\sigma}}_-\hat{\boldsymbol{\sigma}}_+ \\ \hat{\mathbf{C}}^{(\pm)}_{n_xn_y} = C^{(\pm)}_{n_xn_y}\hat{\boldsymbol{\sigma}}_+\hat{\boldsymbol{\sigma}}_- + C^{(\pm)}_{n_x+1,n_y+1}\hat{\boldsymbol{\sigma}}_-\hat{\boldsymbol{\sigma}}_+\end{cases} \tag{35}$$

and the single potential raising operator given by

$$\hat{K}_+^{(\alpha)} = \frac{1}{\sqrt{\hat{\mathcal{H}}_-^{(\alpha)}}}\hat{B}_+^{(\alpha)}, \qquad \alpha = x \text{ or } y. \tag{36}$$

4. Conjugate-coupled system

4.1. Hamiltonian

For the second case we introduce a coupled system the dynamics of which is governed by the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_\xi, \qquad \text{where} \qquad \hat{H}_0 = \hbar\Omega\left\{\left(\hat{A}_x\hat{A}_x^\dagger + \hat{A}_y^\dagger\hat{A}_y\right)\hat{\sigma}_+\hat{\sigma}_- + \left(\hat{A}_x^\dagger\hat{A}_x + \hat{A}_y\hat{A}_y^\dagger\right)\hat{\sigma}_-\hat{\sigma}_+\right\}, \tag{37}$$

and the Hamiltonians for the two kinds of interactions are given by

$$\hat{H}_\xi = \hbar\Delta\hat{\sigma}_3 + \hbar g\begin{cases}\hat{A}_x\hat{A}_y^\dagger\hat{\sigma}_+ + \hat{A}_y\hat{A}_x^\dagger\hat{\sigma}_-, & \xi = \text{U},\\ \hat{A}_x\sqrt{\hat{N}_x\hat{N}_y}\hat{A}_y^\dagger\hat{\sigma}_+ + \hat{A}_y\sqrt{\hat{N}_y\hat{N}_x}\hat{A}_x^\dagger\sigma_-, & \xi = \text{N}.\end{cases} \tag{38}$$

The g and Δ constants have the same meaning as before. In this case if we take harmonic oscillator potentials, we obtain the usual and intensity-dependent versions of the Jaynes–Cummings [13, 19] and of the anti-Jaynes–Cummings models [20] describing a two-level atom interacting non-resonantly with a field $\hat{A}_x$ and with a field $\hat{A}_y$, respectively.

Applying the algebraic formulation presented in section 2, we can write $\hat{H} = \hat{\mathcal{T}}_\pm\hat{h}\hat{\mathcal{T}}_\pm^\dagger$ if we define the parameter translation inclusive operator $\hat{\mathcal{T}}_\pm = \hat{T}_x\hat{\sigma}_+\hat{\sigma}_- \pm \hat{T}_y\hat{\sigma}_-\hat{\sigma}_+$ and decompose the Hamiltonian $\hat{h}$ as $\hat{h} = \hat{h}_0 + \hat{h}_\xi$, where

$$\hat{h}_0 = \hbar\Omega\left\{\left(\hat{\mathcal{H}}_+^{(x)} + \hat{\mathcal{H}}_-^{(y)}\right)\hat{\sigma}_+\hat{\sigma}_- + \left(\hat{\mathcal{H}}_-^{(x)} + \hat{\mathcal{H}}_+^{(y)}\right)\hat{\sigma}_-\hat{\sigma}_+\right\}, \tag{39}$$

and the Hamiltonian $\hat{h}_\xi$ for the two kinds of interaction is given by

$$\hat{h}_\xi = \hbar\Delta\hat{\sigma}_3 \pm \hbar g\begin{cases}\left(\hat{B}_-^{(x)}\hat{B}_+^{(y)}\hat{\sigma}_+ + \hat{B}_-^{(y)}\hat{B}_+^{(x)}\hat{\sigma}_-\right), & \xi = \text{U},\\ \left(\hat{B}_-^{(x)}\sqrt{\hat{N}_x\hat{N}_y}\hat{B}_+^{(y)}\hat{\sigma}_+ + \hat{B}_-^{(y)}\sqrt{\hat{N}_y\hat{N}_x}\hat{B}_+^{(x)}\hat{\sigma}_-\right), & \xi = \text{N}.\end{cases} \tag{40}$$

Again the form of $\hat{H}_0$ permits a freedom of sign choice in the definition of $\hat{\mathcal{T}}_\pm$.

4.2. Model superalgebra

In this case the supercharge operators and their adjoint operators must be defined as $\hat{\mathcal{Q}}_x = \hat{B}_-^{(x)}\hat{\sigma}_+$, $\hat{\mathcal{Q}}_x^\dagger = \hat{B}_+^{(x)}\hat{\sigma}_-$ and $\hat{\mathcal{Q}}_y = \hat{B}_+^{(y)}\hat{\sigma}_+$, $\hat{\mathcal{Q}}_y^\dagger = \hat{B}_-^{(y)}\hat{\sigma}_-$ to permit us to rewrite the Hamiltonian $\hat{h}_0$ in (39) in the same form as in equation (18). With these new supercharge definitions and the form (18) of $\hat{h}_0$, it is straightforward to verify that the commutation and anti-commutation relations (19) which characterize the supersymmetry of the Hamiltonian $\hat{h}_0$ remain valid.

On the other hand we introduce the hybrid supercharge operator and its adjoint operator in the forms $\hat{\mathcal{Q}}_{xy} = \hat{B}_-^{(x)}\hat{B}_+^{(y)}\hat{\sigma}_+$ and $\hat{\mathcal{Q}}_{xy}^\dagger = \hat{B}_+^{(x)}\hat{B}_-^{(y)}\hat{\sigma}_-$ for the ξ = N interaction case and in the forms $\hat{\mathcal{Q}}_{xy} = \hat{B}_-^{(x)}\sqrt{\hat{N}_x\hat{N}_y}\hat{B}_+^{(y)}\hat{\sigma}_+$ and $\hat{\mathcal{Q}}_{xy}^\dagger = \hat{B}_-^{(y)}\sqrt{\hat{N}_y\hat{N}_x}\hat{B}_+^{(x)}\hat{\sigma}_-$ for the ξ = N interaction case to permit us to rewrite the Hamiltonian $\hat{h}_\xi$ in the same form as in equation (20). Obviously in both cases the additional commutation and anti-commutation relations (21) are still valid. The action of the hybrid supercharge operator $\hat{\mathcal{Q}}_{xy}$ and its adjoint operator $\hat{\mathcal{Q}}_{xy}^\dagger$ in the *heating* and *cooling* process of the conjugate-coupled system is schematically illustrated in figure 2.

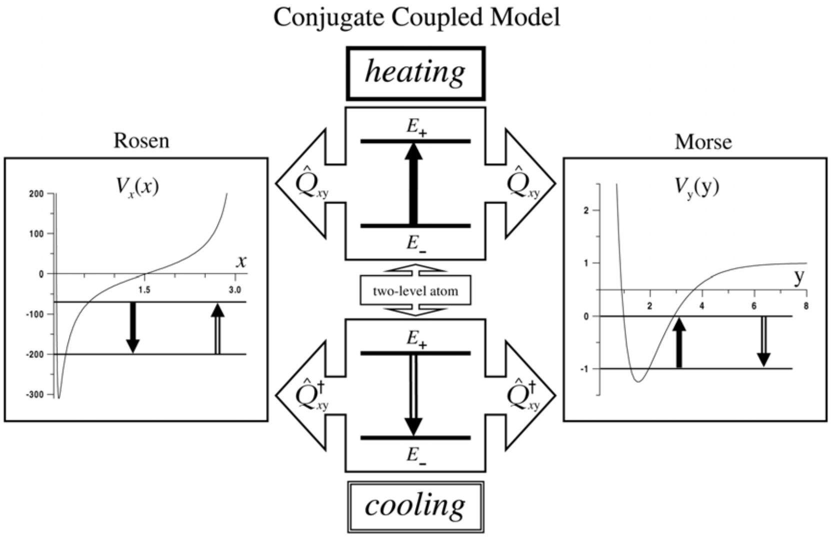

Figure 2. Same as figure 1 for the case of the conjugate-coupled model.

4.3. Eigenstates and eigenvalues

Since $[\hat{H}_0, \hat{H}_\xi] = 0$, it is possible to find a common set of eigenstates for them. Thus, introducing the dressed state

$$|\Psi^{(\pm)}\rangle = \hat{\mathcal{T}}_\pm\{C^{(\pm)}_{n_x n_y}|n_x\rangle_x|n_y\rangle_y|+\rangle + C^{(\pm)}_{k_x k_y}|k_x\rangle_x|k_y\rangle_y|-\rangle\}$$
$$= \hat{T}_x C^{(\pm)}_{n_x n_y}|n_x\rangle_x|n_y\rangle_y|+\rangle \pm \hat{T}_y C^{(\pm)}_{k_x k_y}|k_x\rangle_x|k_y\rangle_y|-\rangle \tag{41}$$

in the eigenvalue equation $\hat{H}_0|\Psi\rangle = E^{(0)}|\Psi\rangle$ and using equations (7), (39) and (41), we obtain the system of equations

$$\begin{cases}\{\hbar\Omega\hat{T}_x[e^{(x)}_{n_x} + R_x(a^{(x)}_0) + e^{(y)}_{n_y}]\hat{T}^\dagger_x\}\hat{T}_x C^{(\pm)}_{n_x n_y}|n_x\rangle_x|n_y\rangle_y = E^{(0)}\hat{T}_x C^{(\pm)}_{n_x n_y}|n_x\rangle_x|n_y\rangle_y \\ \{\hbar\Omega\hat{T}_y[e^{(x)}_{k_x} + e^{(y)}_{k_y} + R_y(a^{(y)}_0)]\hat{T}^\dagger_y\}\hat{T}_y C^{(\pm)}_{k_x k_y}|k_x\rangle_x|k_y\rangle_y = E^{(0)}\hat{T}_y C^{(\pm)}_{k_x k_y}|k_x\rangle_x|k_y\rangle_y\end{cases} \tag{42}$$

the solution of which implies that we have $k_x = n_x + 1$ and $k_y = n_y - 1$. Thus, after we redefine $n_y \to n_y + 1$, we find the eigenstate

$$\left|\Psi^{(\pm)}_{n_x n_y}\right\rangle = \hat{T}_x C^{(\pm)}_{n_x,n_y+1}|n_x\rangle_x|n_y+1\rangle_y|+\rangle \pm \hat{T}_y C^{(\pm)}_{n_x+1,n_y}|n_x+1\rangle_x|n_y\rangle_y|-\rangle \tag{43}$$

and the associated eigenvalue $E^{(0)}_{n_x n_y}$ with the same form obtained for the first coupled system studied.

In this case the two-dimensional spinor representation of the eigenstates (43) of the coupled system can be written in terms of the coupling potential eigenfunctions $\psi^{(x)}_\mu(x)$ and $\psi^{(y)}_\nu(y)$ as

$$\Psi^{(\pm)}_{n_x n_y}(x, y) = \left\langle x, y\middle|\Psi^{(\pm)}_{n_x n_y}\right\rangle = \begin{bmatrix}\hat{T}_x C^{(\pm)}_{n_x,n_y+1}\psi_{n_x,n_y+1}(x, y) \\ \pm\hat{T}_y C^{(\pm)}_{n_x+1,n_y}\psi_{n_x+1,n_y}(x, y)\end{bmatrix}, \tag{44}$$

where $\psi_{\mu\nu}(x, y) \equiv \langle x|\mu\rangle_x\langle y|\nu\rangle_y = \psi^{(x)}_\mu(x)\psi^{(y)}_\nu(y)$. Note that the orthonormality of the eigenstates $\left|\Psi^{(\pm)}_{n_x n_y}\right\rangle$ implies the following relations among the C-coefficients:

$$\left[C^{(\pm)}_{n_x,n_y+1}\right]^2 + \left[C^{(\pm)}_{n_x+1,n_y}\right]^2 = 1 \qquad \text{and} \qquad C^{(\pm)}_{n_x,n_y+1}C^{(\mp)}_{n_x,n_y+1} - C^{(\pm)}_{n_x+1,n_y}C^{(\mp)}_{n_x+1,n_y} = 0. \tag{45}$$

To determine the eigenvalues of $\hat{H}_\xi$, we need to resolve the equation $\hat{H}_\xi \left|\Psi^{(\pm)}_{n_x n_y}\right\rangle = \mathcal{E}^{(\pm)}_{\xi n_x n_y}\left|\Psi^{(\pm)}_{n_x n_y}\right\rangle$. Using equations (38), (40) and (43) in this eigenvalue equation for the $\xi = \mathrm{U}$ interaction case, we obtain the system of equations

$$\begin{cases} \hbar\Delta\hat{T}_x C^{(\pm)}_{n_x,n_y+1}|n_x\rangle_x|n_y+1\rangle_y \pm \hbar g\hat{T}_x\hat{B}^{(x)}_-\hat{B}^{(y)}_+ C^{(\pm)}_{n_x+1,n_y}|n_x+1\rangle_x|n_y\rangle_y \\ \quad = \mathcal{E}^{(\pm)}_{\mathrm{U}n_x n_y}\hat{T}_x C^{(\pm)}_{n_x,n_y+1}|n_x\rangle_x|n_y+1\rangle_y \\ \hbar g\hat{T}_y\hat{B}^{(y)}_-\hat{B}^{(x)}_+ C^{(\pm)}_{n_x,n_y+1}|n_x\rangle_x|n_y+1\rangle_y \mp \hbar\Delta\hat{T}_y C^{(\pm)}_{n_x+1,n_y}|n_x+1\rangle_x|n_y\rangle_y \\ \quad = \pm\mathcal{E}^{(\pm)}_{\mathrm{U}n_x n_y}\hat{T}_y C^{(\pm)}_{n_x+1,n_y}|n_x+1\rangle_x|n_y\rangle_y \end{cases} \tag{46}$$

and by using relations (10) and (11), we find the eigenvalue $\mathcal{E}^{(\pm)}_{\mathrm{u}n_x n_y}$ in the same form as in (29) and the additional coefficient relation

$$\hat{T}_x C^{(\pm)}_{n_x,n_y+1}\hat{T}^\dagger_x = \left\{\sqrt{1+\upsilon^2_{n_x n_y}} \mp \upsilon_{n_x n_y}\right\}\hat{T}_y C^{(\pm)}_{n_x+1,n_y}\hat{T}^\dagger_y, \tag{47}$$

where the $\upsilon_{n_x n_y}$ factor is still given by equation (30).

Following the same steps for the $\xi = \mathrm{N}$ interaction case with equations (38), (40) and (43) in the eigenvalue equation, we obtain the system of equations

$$\begin{cases} \hbar\Delta\hat{T}_x C^{(\pm)}_{n_x,n_y+1}|n_x\rangle_x|n_y+1\rangle_y \pm \hbar g\hat{T}_x\hat{B}^{(x)}_-\sqrt{\hat{N}_x\hat{N}_y}\hat{B}^{(y)}_+ C^{(\pm)}_{n_x+1,n_y}|n_x+1\rangle_x|n_y\rangle_y \\ \quad = \mathcal{E}^{(\pm)}_{\mathrm{N}n_x n_y}\hat{T}_x C^{(\pm)}_{n_x,n_y+1}|n_x\rangle_x|n_y+1\rangle_y \\ \hbar g\hat{T}_y\hat{B}^{(y)}_-\sqrt{\hat{N}_y\hat{N}_x}\hat{B}^{(x)}_+ C^{(\pm)}_{n_x,n_y+1}|n_x\rangle_x|n_y+1\rangle_y \mp \hbar\Delta\hat{T}_y C^{(\pm)}_{n_x+1,n_y}|n_x+1\rangle_x|n_y\rangle_y \\ \quad = \pm\mathcal{E}^{(\pm)}_{\mathrm{N}n_x n_y}\hat{T}_y C^{(\pm)}_{n_x+1,n_y}|n_x+1\rangle_x|n_y\rangle_y \end{cases} \tag{48}$$

the solution of which gives the eigenvalue $\mathcal{E}^{(\pm)}_{\mathrm{N}n_x n_y}$ in the same form as in (32) and an additional relation among the C-coefficients with the same form (47) but with the factor $\upsilon_{n_x n_y}$ now given by $\upsilon_{n_x n_y} = \Delta/\left(g e^{(x)}_{n_x+1} e^{(y)}_{n_y+1}\right)$.

To conclude this part, we note that the excited states $\left|\Psi^{(\pm)}_{n_x n_y}\right\rangle$ of the coupled system can be obtained from the two-dimensional spinor state

$$\left|\Psi^{(\pm)}_{00}\right\rangle = \begin{bmatrix} \hat{T}_x C^{(\pm)}_{01}|0\rangle_x|1\rangle_y \\ \pm\hat{T}_y C^{(\pm)}_{10}|1\rangle_x|0\rangle_y \end{bmatrix} \tag{49}$$

by using the expression $\left|\Psi^{(\pm)}_{n_x n_y}\right\rangle = \hat{\mathbf{K}}^{(\pm)}_{n_x n_y}\left|\Psi^{(\pm)}_{00}\right\rangle$, where the raising operator matrix $\hat{\mathbf{K}}^{(\pm)}_{n_x n_y}$ is given by expression (35) with the parameter translation matrix $\hat{\mathbf{T}}_\pm$ and the coefficient matrix $\hat{\mathbf{C}}^{(\pm)}_{n_x n_y}$ now defined as

$$\hat{\mathbf{T}}_\pm = \hat{T}_x\hat{\boldsymbol{\sigma}}_+\hat{\boldsymbol{\sigma}}_- \pm \hat{T}_y\hat{\boldsymbol{\sigma}}_-\hat{\boldsymbol{\sigma}}_+ \quad \text{and} \quad \hat{\mathbf{C}}^{(\pm)}_{n_x n_y} = C^{(\pm)}_{n_x,n_y+1}\hat{\boldsymbol{\sigma}}_+\hat{\boldsymbol{\sigma}}_- + C^{(\pm)}_{n_x+1,n_y}\hat{\boldsymbol{\sigma}}_-\hat{\boldsymbol{\sigma}}_+. \tag{50}$$

The results obtained with the two models (direct and conjugate) thus lead to the conclusion that, despite their differences, both models share the same spectra of eigenvalues $E^{(\pm)}_{n_x n_y}$ but have different sets of eigenstates $\left|\Psi^{(\pm)}_{n_x n_y}\right\rangle$.

5. Examples

To illustrate how our general results can be applied in specific cases, we work out in this section three examples of shape-invariant potential pairs: (i) a pair of harmonic oscillator potentials, (ii) a harmonic oscillator and a Morse potentials and (iii) a harmonic oscillator and a self-similar potential. Our intention with the first example is to rederive quantum optics models. Our motivation with the other two examples is to investigate the effects of an asymmetric coupling potential and to evaluate the influence of a coupling potential, where parameters a_n are related by a scaling in the coupled system eigenvalues and eigenfunctions.

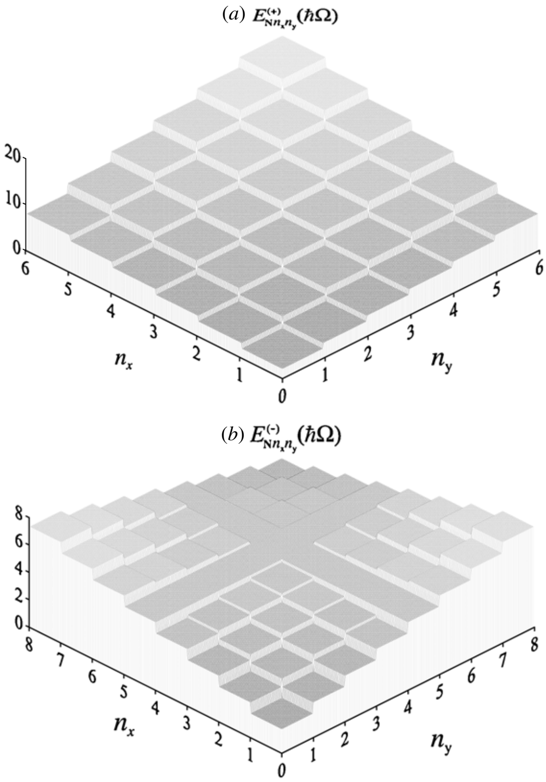

Figure 3. Three-dimensional graphics of the eigenvalues $E^{(+)}_{\mathrm{N}n_xn_y}$ and $E^{(-)}_{\mathrm{N}n_xn_y}$ in terms of the quantum numbers n_x and n_y, respectively. The set of parameters used is presented in the text.

5.1. Two harmonic oscillator coupling potentials

We start with this example because it is the simplest among the shape-invariant coupling potentials and describes interaction of matter, represented by a two-level atom, with a quantized two-mode electromagnetic field, represented by the harmonic oscillator bosonic operators $\hat{A}_x$ and $\hat{A}_y$. The partner potentials (3) for these systems are obtained with the superpotentials $W_x\left(x, a_1^{(x)}\right) = \sqrt{\hbar\Omega}\left(a_1^{(x)}x + \zeta_x\right)$ and $W_y\left(y, a_1^{(y)}\right) = \sqrt{\hbar\Omega}\left(a_1^{(y)}y + \zeta_y\right)$, where $a_1^{(x,y)}$ and $\zeta_{x,y}$ are real constants, while the remainders [1] in the shape-invariant condition (4) are given by $R_\alpha\left(a_n^{(\alpha)}\right) = \eta\left(a_n^{(\alpha)} + a_{n+1}^{(\alpha)}\right)$, where $\eta = \sqrt{\hbar/(2m\Omega)}$. Taking into account that the parameters for this potential are related by $a_1^{(\alpha)} = a_2^{(\alpha)} = \cdots = a_n^{(\alpha)}$, the remainders can be written as $R_\alpha\left(a_n^{(\alpha)}\right) = \gamma_\alpha$, with $\gamma_\alpha = 2\eta a_1^{(\alpha)}$, and thus

$$e_n^{(\alpha)} = n\gamma_\alpha \qquad \text{with} \quad \alpha = x \text{ or } y. \tag{51}$$

Under these conditions the coupled system energy eigenvalues $E^{(\pm)}_{\xi n_xn_y}$ in units of $\hbar\Omega$, obtained when we use equation (51) in (33), are given by the expression

$$E^{(\pm)}_{\xi n_xn_y}/\hbar\Omega = \{\gamma_x(n_x+1) + \gamma_y(n_y+1)\} \pm \epsilon \begin{cases} \sqrt{\gamma_x\gamma_y(n_x+1)(n_y+1) + \delta^2}, & \text{if} \quad \xi = \mathrm{U}, \\ \sqrt{\{\gamma_x\gamma_y(n_x+1)(n_y+1)\}^2 + \delta^2}, & \text{if} \quad \xi = \mathrm{N}. \end{cases} \tag{52}$$

Figure 3 shows two-dimensional representation of the coupled system energy eigenvalues $E^{(\pm)}_{\mathrm{N}n_xn_y}$ in units of $\hbar\Omega$ obtained when we use equation (52) and assume the values $\gamma_x = \gamma_y =$

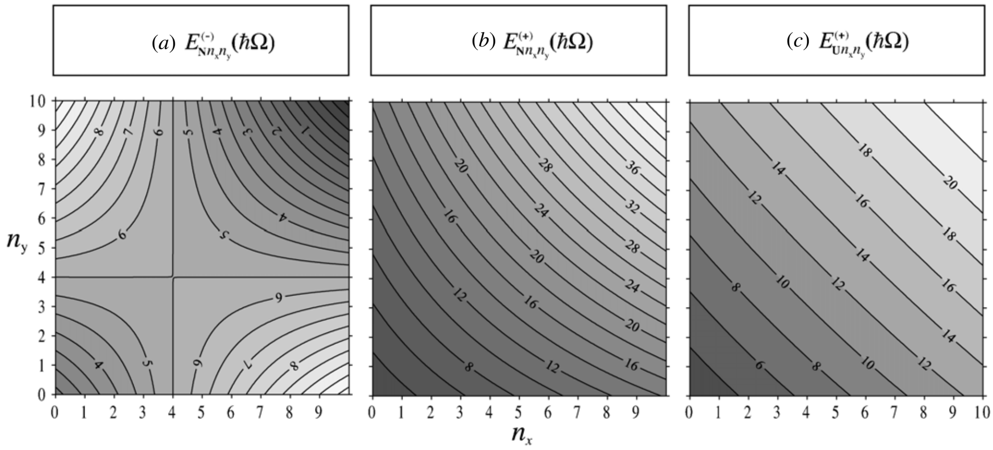

Figure 4. Smooth level curve of the eigenvalues of the coupled system with two harmonic oscillator potentials projected in a $n_x \times n_y$ plane. Figures (*a*) and (*b*) show, respectively, $E^{(-)}_{Nn_xn_y}$ and $E^{(+)}_{Nn_xn_y}$ while figure (*c*) shows $E^{(+)}_{Un_xn_y}$ calculated with the same strengths.

1.0, $\epsilon = 0.2$ and $\delta = 1.0$. Before we discuss the behaviour of the eigenvalues of the coupled system, it is interesting to observe that the general form of the coupling term $\mathcal{E}^{(\pm)}_{\xi n_xn_y}$ represents a particular case, with a δ^2 shift, of the Cobb–Douglas production function in economics that relates productivity to labour and capital [21]. Figures 3(*a*) and (*b*) express the behaviour characteristic of that function. For $E^{(+)}_{Nn_xn_y}$ it is evident that the coupling increases the value of $E^{(\pm)}_{Nn_xn_y}$ along the principal diagonal defined from (0,0) to (8,8). However the behaviour of $E^{(-)}_{Nn_xn_y}$ is certainly more interesting since, in this case, the competition between the terms $E^{(0)}_{n_xn_y}$ and $\mathcal{E}^{(-)}_{Nn_xn_y}$ in $E^{(-)}_{Nn_xn_y}$ create a saddle point in (4,4). Therefore, if we imagine standing at this saddle point, $E^{(-)}_{Nn_xn_y}$ presents a minimum value when we follow the diagonal defined from (8,0) to (0,8) and slopes up away from this point. Along the principal diagonal $E^{(-)}_{Nn_xn_y}$ presents a maximum value in the saddle point and slopes down away from this point. On the other hand, using (52) we can show that along the straight lines $(4, n_y)$ and $(n_x, 4)$, the eigenvalues have the almost constant value

$$E^{(-)}_{\mathrm{N},n_x,4} = E^{(-)}_{\mathrm{N},4,n_y} \approx \left\{5 - \frac{1}{10[n_{x,y}+1]}\right\}\hbar\Omega \approx 5\hbar\Omega. \tag{53}$$

Obviously this value, the position of those lines as well as the saddle point position are dependent on the parameter set used.

We show in figure 4 the smooth level curves of $E^{(\pm)}_{\xi n_xn_y}$ in units of $\hbar\Omega$, obtained with equation (52) and parameter values presented above, projected in a $n_x \times n_y$ plane assuming continuous real values for these quantum numbers. To help the visualization, we present each energy level with a different shading (darker regions are related to lower energy values). To contrast different possibilities, we present together such curves for $E^{(-)}_{Nn_xn_y}$ in (*a*), $E^{(+)}_{Nn_xn_y}$ in (*b*) and $E^{(+)}_{Un_xn_y}$ in (*c*). Because of the isotropy of the shape-invariant potentials, we observe that the three figures present a symmetric behaviour along the principal diagonal. We can compare the behaviour of the eigenvalues obtained with the two different kinds of interaction. The enhancement effect on the coupling term $\mathcal{E}^{(+)}_{Nn_xn_y}$ compared with the usual interaction term

$\mathcal{E}^{(+)}_{\mathrm{U}n_x n_y}$ is evident if we compare figures 4(*b*) and (*c*). We also observe in figure 4(*a*) the saddle point in (4,4) as well as the almost isoenergetic lines $(4, n_y)$ and $(n_x, 4)$.

The spinor elements of the eigenfunctions $\Psi^{(\pm)}_{n_x n_y}(x, y)$ in (26) and (44), for the direct and the conjugate models, respectively, are given by $\psi_{\mu\nu}(x, y) = \psi_\mu\left[a_1^{(x)}x + \zeta_x\right]\psi_\nu\left[a_1^{(y)}y + \zeta_y\right]$, where $\psi_n(u) = e^{-u^2/2}\,\mathrm{H}_n(u)$ and $\mathrm{H}_n(u)$ are the Hermite polynomials [24].

5.2. Harmonic oscillator plus Morse coupling potentials

The one-dimensional Morse potential, originally introduced as a useful model for the diatomic molecules [22], has been widely used in many areas of physics to study physics phenomena such as molecular vibrations, laser chemistry and, in particular, chemical bonds. Anharmonicities and dissociation effects, which may arise in a more realistic physical situation, are better represented using a Morse potential. Therefore the partner potentials (3) for this second example are obtained with the superpotentials $W_x\left(x, a_1^{(x)}\right) = \sqrt{\hbar\Omega}\left(a_1^{(x)}x + \zeta\right)$ for the harmonic oscillator and $W_y\left(y, a_1^{(y)}\right) = \sqrt{\hbar\Omega}\left\{a_1^{(y)} - e^{-\varrho y}\right\}$ for the Morse potential [1], where $a_1^{(x,y)}$, ζ and ϱ are real constants.

The remainders [1] in the shape-invariant condition (4) for the Morse potential case are given by $R\left(a_n^{(y)}\right) = \eta_y\left(2a_n^{(y)} - \eta_y\right)$, with the potential parameters related by $a_{n+1}^{(y)} = a_n^{(y)} - \eta_y$, where $\eta_y = \sqrt{\hbar/(2m\Omega)}\varrho$. Using these results in (9), we can prove that the energy eigenstates are given by

$$e_n^{(y)} = \eta_y^2 n(2\zeta_y - n), \qquad \text{with} \quad n \leqslant \zeta_y \equiv a_1^{(y)}\big/\eta_y. \tag{54}$$

Under these conditions the coupled system energy eigenvalues $E^{(\pm)}_{\xi n_x n_y}$ in units of $\hbar\Omega$, obtained when we use (51) for $\alpha = x$ and (54) for $\alpha = y$ in equation (33), are given by the expression

$$E^{(\pm)}_{\xi n_x n_y}\big/\hbar\Omega = \left\{\gamma_x(n_x+1) + \eta_y^2(n_y+1)(2\zeta_y - n_y - 1)\right\}$$
$$\pm\,\epsilon\begin{cases}\sqrt{\gamma_x\eta_y^2(n_x+1)(n_y+1)(2\zeta_y - n_y - 1) + \delta^2}, & \text{if} \quad \xi = \mathrm{U},\\ \sqrt{\left\{\gamma_x\eta_y^2(n_x+1)(n_y+1)(2\zeta_y - n_y - 1)\right\}^2 + \delta^2}, & \text{if} \quad \xi = \mathrm{N}.\end{cases} \tag{55}$$

Figure 5 is similar to figure 4, but with a Morse potential and harmonic oscillator pair, obtained when we use (55) and assume the parameter values $\gamma_x = 1.0$, $\epsilon = 0.2$, $\delta = 1.0$, $a_1^{(y)} = 10\eta_y$ and $\eta_y = 1$. Looking at this figure, it is evident that the anisotropy of the coupling potential system is manifested by the absence of symmetry lines for the level eigenvalue curves. Comparing figures 5(*b*) and (*c*), one can conclude that the coupling term $\mathcal{E}^{(+)}_{\xi n_x n_y}$ is dominant for the intensity-dependent interaction case. We observe in figure 5(*a*) that the predominance of the coupling term $\mathcal{E}^{(-)}_{\mathrm{N}n_x n_y}$ over the term $E^{(0)}_{n_x n_y}$ in $E^{(-)}_{\mathrm{N}n_x n_y}$ for high values of n_x and n_y makes the eigenvalues assume negative values in this region. Obviously all these characteristic facts of $E^{(\pm)}_{\xi n_x n_y}$ can be justified by looking at expression (55).

The spinor elements of the coupled system eigenfunctions $\Psi^{(\pm)}_{n_x n_y}(x, y)$ in (26) and (44), for the direct and conjugate models, respectively, are given by $\psi_{\mu\nu}(x, y) = \psi_\mu^{(x)}\left[a_1^{(x)}x + \zeta_x\right]\psi_\nu^{(y)}\left[2\zeta_y e^{-\varrho y}\right]$ where $\psi_n^{(x)}(u) = e^{-u^2/2}\,\mathrm{H}_n(u)$ while $\psi_n^{(y)}(u) = e^{-u/2}u^{\lambda/2}\,\mathrm{L}_n^{\lambda}(u)$ with $\lambda = 2\zeta_y - 2n - 1$ and being $\mathrm{L}_n^{\lambda}(u)$ the associated Laguerre polynomials [24].

5.3. Harmonic oscillator + self-similar coupling potentials

One class of shape-invariant potentials is given by an infinite chain of reflectionless potentials $V_\pm^{(k)}(y)$, $(k = 0, 1, 2, \ldots)$, the associated superpotentials $W_k(y)$ for which satisfy the self-

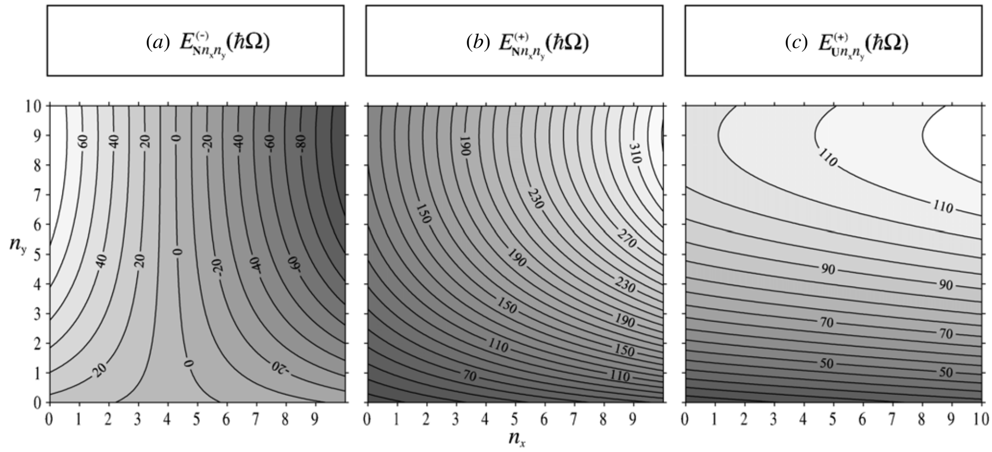

Figure 5. Same as figure 4 calculated for a pair with a harmonic oscillator plus a Morse potential. The set of parameters used is presented in the text.

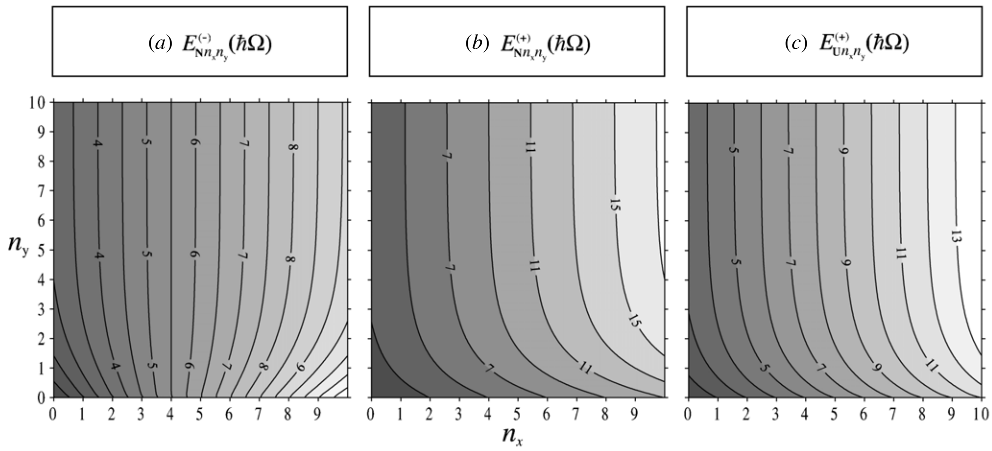

Figure 6. Same as figure 4 calculated for a pair with a harmonic oscillator plus a self-similar potential. The set of parameters used is presented in the text.

similar ansatz $W_k(y) = q^k W(q^k y)$, with $0 < q < 1$. These sets of partner potentials $V_{\pm}^{(k)}(y)$, also called self-similar potentials [23], have an infinite number of bound states and their parameters related by a scaling: $a_{n_y}^{(y)} = q^{n_y-1}a_1^{(y)}, \quad \forall n_y \in Z$. The self-similar potentials can be considered as quantum deformations of the multisoliton solutions corresponding to the Rosen–Morse potential. Indeed working with this kind of potential it is possible to get the Rosen–Morse, harmonic oscillator and Pöschl–Teller potentials as limiting cases [23]. Shape invariance of self-similar potentials was studied in detail in [15]. In the simplest case studied, the remainder of equation (4) is given by $R_y\big(a_1^{(y)}\big) = ca_1^{(y)}$, where c is a constant. Using this result in (9), we can prove that

$$e_{n_y}^{(y)} = \left(\frac{1-q^{n_y}}{1-q}\right) R_y\big(a_1^{(y)}\big). \tag{56}$$

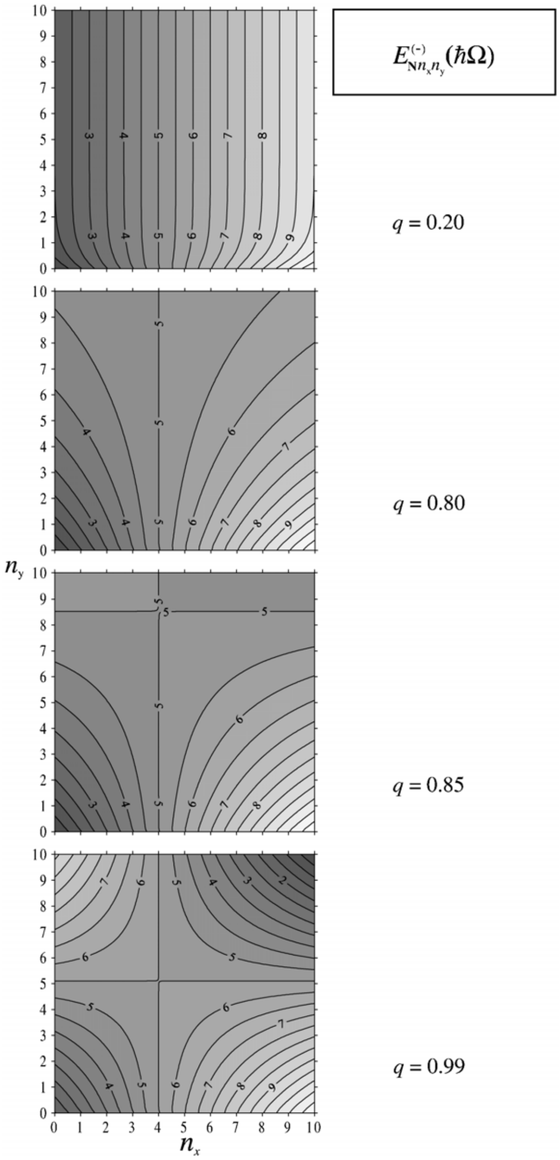

Figure 7. Smooth level curve of the eigenvalues of the coupled system $E^{(-)}_{\mathrm{N}n_x n_y}$ involving a harmonic oscillator and a self-similar potential calculated for the scaling parameter values $q = 0.20, 0.80, 0.85$ and 0.99. The values of the other constants are the same as used before.

Under these conditions the coupled system energy eigenvalues $E^{(\pm)}_{\xi n_x n_y}$ in units of $\hbar\Omega$, obtained when we use (51) for $\alpha = x$ and (56) for $\alpha = y$ in equation (33), are given by the expression

$$E^{(\pm)}_{\xi n_x n_y}\big/\hbar\Omega = \{\gamma_x(n_x+1) + \kappa_y(1-q^{n_y+1})\} \pm \epsilon \begin{cases} \sqrt{\gamma_x\kappa_y(n_x+1)(1-q^{n_y+1}) + \delta^2}, & \text{if} \quad \xi = \mathrm{U}, \\ \sqrt{\{\gamma_x\kappa_y(n_x+1)(1-q^{n_y+1})\}^2 + \delta^2}, & \text{if} \quad \xi = \mathrm{N}, \end{cases} \tag{57}$$

where $\kappa_y = R_y\big(a_1^{(y)}\big)\big/(1-q)$.

Figure 6 is the version of figure 4 for the smooth level curves projected in a $n_x \times n_y$ plane of the coupled system eigenvalues $E^{(\pm)}_{\xi n_x n_y}$ in $\hbar\Omega$ units, obtained when we use (57) and the set of constant values $\gamma_x = 1.0$, $\epsilon = 0.2$, $\delta = 1.0$, $R\big(a_1^{(y)}\big) = 1.0$ and $q = 0.5$. This figure shows that the anisotropy of the coupling potential system is characterized by the absence of symmetry lines for the eigenvalue curves. Besides that the position of the level curves almost parallel to the n_y-axis for high values of n_y observed in all figures expresses the weak influence of the coupling potential $V_y^{(\pm)}(y)$ on the eigenvalue $E^{(\pm)}_{\xi n_x n_y}$ which result, at the end, in a saturation effect. Only for small values of the quantum numbers n_x and n_y, the two potentials have a comparable impact on the $E^{(\pm)}_{\xi n_x n_y}$ values. As in the other two examples, comparing figures 6(*b*) and (*c*), it is possible to verify the enhancement effect on the coupling term $\mathcal{E}^{(+)}_{\mathrm{N}n_x n_y}$ for the intensity-dependent interaction when compared with the usual interaction case $\mathcal{E}^{(+)}_{\mathrm{U}n_x n_y}$.

To visualize the influence of the scaling parameter q of the self-similar potential on the eigenvalues of the coupled system, we plot in figure 7 the smooth level curves of $E^{(-)}_{\mathrm{N}n_x n_y}$ in $\hbar\Omega$ units, obtained with equation (57), projected on a $n_x \times n_y$ plane for $q = 0.20, 0.80, 0.85$ and 0.99. We assume the same values for the other parameters. We observe that as the scaling parameter q increases the saddle point moves closer to the (4,4) position, bringing together the almost isoenergetic straight lines of $E^{(-)}_{\mathrm{N}n_x,n_y} \approx 5\hbar\Omega$. It is evident from this figure that in the limit $q \to 1$, we reproduce the results obtained in the first example involving two harmonic oscillator coupling potentials. This is easy to understand since we note that

$$\begin{aligned} \lim_{q\to 1} e^{(y)}_{n_y} &= \lim_{q\to 1}\left(\frac{1-q^{n_y}}{1-q}\right) R_y\big(a_1^{(y)}\big) \\ &= n_y R_y\big(a_1^{(y)}\big) \end{aligned} \tag{58}$$

that corresponds to equation (51) if we identify $\gamma_y \to R_y\big(a_1^{(y)}\big)$.

6. Conclusions

Exactly soluble and fully quantum-mechanical models are rare. In this paper we introduced a class of bound-state problems which represent a two-level atom coupled with a two-dimensional supersymmetric and shape-invariant potential system. This is a non-trivial coupled-channels' problem which may find applications in molecular, atomic and nuclear physics. Taking into account two possible coupled models (direct- and conjugate-coupled models) and two forms of coupling interaction (constant and intensity dependent), we obtained the eigenvalues and the eigenstates of the system and discussed the differences and similarities presented by each model and the two different kinds of couplings. We studied the behaviour of these eigenvalues problem quantities for three different pairs of shape-invariant potentials (two harmonic oscillators, harmonic oscillator + Morse and harmonic oscillator + self-similar potentials). For pairs of shape-invariant potentials involving other than the harmonic oscillator, the eigenvalues present asymmetric behaviour in relation to the quantum numbers n_x and n_y, basically determined by the coupling eigenvalue term $\mathcal{E}^{(-)}_{\xi n_x n_y}$ the form of which is a particular case of the production function of Cobb–Douglas in economics.

In the next paper of this series [25], we investigate the dynamics of the models we introduced.

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